

Determinacy in $L(\mathbb{R}, \mu)$

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Abstract

Assume $V = L(\mathbb{R}, \mu) \models ZF + DC + \Theta > \omega_2 + \mu$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$. We analyze what sets of reals are determined and in fact show that $L(\mathbb{R}, \mu) \models AD$. This arguably gives the most optimal characterization of AD in $L(\mathbb{R}, \mu)$. As a consequence of this analysis, we obtain the equiconsistency of the theories: “ $ZFC + \text{There are } \omega^2 \text{ Woodin cardinals}$ ” and “ $ZF + DC + \Theta > \omega_2 + \text{There is a normal fine measure on } \mathcal{P}_{\omega_1}(\mathbb{R})$ ”.

1 Introduction

It is well-known that the existence of an $L(\mathbb{R}, \mu)^1$ that satisfies $ZF + DC + \mu$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})^2$ is equiconsistent with that of a measurable cardinal. The model $L(\mathbb{R}, \mu)$ obtained from standard proofs of the equiconsistency satisfies $\Theta^3 = \omega_2$ and hence fails to satisfy AD . So it is natural to consider the situations where $L(\mathbb{R}, \mu) \models \Theta > \omega_2$ and try to understand how much determinacy holds in this model.

To analyze the sets of reals that are determined, we run the core model induction in a

¹By $L(\mathbb{R}, \mu)$ we mean the model constructed from the reals and using μ as a predicate. We will also use the notation $L(\mathbb{R})[\mu]$ and $L_\alpha(\mathbb{R})[\mu]$ in various places in the paper.

²A measure μ on $\mathcal{P}_{\omega_1}(\mathbb{R})$ is fine if for all $x \in \mathbb{R}$, $\mu(\{\sigma \in \mathcal{P}_{\omega_1}(\mathbb{R}) \mid x \in \sigma\}) = 1$. μ is normal if for all functions $F : \mathcal{P}_{\omega_1}(\mathbb{R}) \rightarrow \mathcal{P}_{\omega_1}(\mathbb{R})$ such that $\mu(\{\sigma \mid F(\sigma) \subseteq \sigma\}) = 1$, there is an $x \in \mathbb{R}$ such that $\mu(\{\sigma \mid x \in F(\sigma)\}) = 1$.

³ Θ is the sup of all α such that there is a surjection from \mathbb{R} onto α

certain submodel of V that agrees with V on all bounded subsets of Θ . This model will be defined in the next section. What we'll show is that $K(\mathbb{R}) \models AD^+$ where

$$K(\mathbb{R}) = L(\bigcup\{\mathcal{M} \mid \mathcal{M} \text{ is an } \mathbb{R}\text{-premouse, } \rho(\mathcal{M}) = \mathbb{R}, \text{ and } \mathcal{M} \text{ is countably iterable}^4\}).$$

We will then show $\Theta^{K(\mathbb{R})} = \Theta$ by an argument like that in Chapter 7 of [7]. Finally, in section 4, we'll prove that

$$\mathcal{P}(\mathbb{R}) \cap K(\mathbb{R}) = \mathcal{P}(\mathbb{R}),$$

which implies $L(\mathbb{R}, \mu) \models AD$. Woodin has shown the following.

Theorem 1.1. *Suppose $L(\mathbb{R}, \mu) \models AD + \mu$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$. Then $L(\mathbb{R}, \mu) \models AD^+ + \mu$ is unique.*

Using Theorem 1.1, we state the main result of this paper, which gives a surprising characterization of AD^+ in these models.

Theorem 1.2. *Suppose $V = L(\mathbb{R}, \mu) \models ZF + DC + \Theta > \omega_2 + \mu$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$. Then $L(\mathbb{R}, \mu) \models AD^+ + \mu$ is unique.*

The equiconsistency result we get from this analysis is the following.

Theorem 1.3. *The following theories are equiconsistent.*

1. $ZFC + \text{There are } \omega^2 \text{ Woodin cardinals.}$
2. $ZF + DC + AD^+ + \text{There is a normal fine measure on } \mathcal{P}_{\omega_1}(\mathbb{R}).$
3. $ZF + DC + \Theta > \omega_2 + \text{There is a normal fine measure on } \mathcal{P}_{\omega_1}(\mathbb{R}).$

Proof. The equiconsistency of (1) and (2) is a theorem of Woodin (see [15] for more information). Theorem 1.2 immediately implies the equiconsistency of (2) and (3). \square

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⁴An \mathbb{R} -premouse \mathcal{M} is countably iterable if any countable hull of \mathcal{M} is $\omega_1 + 1$ iterable.

2 The core model induction through $K(\mathbb{R})$

First note that we cannot well-order the reals hence full AC fails in this model. Secondly, ω_1 is regular; this follows from DC . Now μ induces a countably complete nonprincipal ultrafilter on ω_1 ; hence, ω_1 is a measurable cardinal. DC also implies that $\text{cof}(\omega_2) > \omega$.

Lemma 2.1. Θ is a regular cardinal.

Proof. Suppose not. Let $f : \mathbb{R} \rightarrow \Theta$ be a cofinal map. Then there is an $x \in \mathbb{R}$ such that f is $OD(\mu, x)$. For each $\alpha < \Theta$, there is a surjection $g_\alpha : \mathbb{R} \rightarrow \alpha$ such that g_α is $OD(\mu)$ (we may take g_α to be the least such). We can get such a g_α because we can “average over the reals.” Now define a surjection $g : \mathbb{R} \rightarrow \Theta$ as follows

$$g(y) = g_{f(y_0)}(y_1) \text{ where } y = \langle y_0, y_1 \rangle.$$

It’s easy to see that g is a surjection. But this is a contradiction. \square

Lemma 2.2. ω_1 is inaccessible in any (transitive) inner model of choice containing ω_1 .

Proof. This is easy. Let N be such a model. Since $P = L(N, \mu)$ is also a choice model and ω_1 is measurable in P , hence ω_1 is inaccessible in P . This gives ω_1 is inaccessible in N . \square

Next, we define two key models that we’ll use for our core model induction. Let

$$M = \prod_{\sigma \in \mathcal{P}_{\omega_1}(\mathbb{R})} M_\sigma \text{ where } M_\sigma = HOD_{\sigma \cup \{\sigma, \mu\}}$$

and,

$$H = \prod_{\sigma \in \mathcal{P}_{\omega_1}(\mathbb{R})} H_\sigma \text{ where } H_\sigma = HOD_{\{\sigma, \mu\}}.$$

Lemma 2.3. Lós theorem holds for both of the ultraproducts defined above.

Proof. We do this for the first ultraproduct. The proof is by induction on the complexity of formulas. It’s enough to show the following. Suppose $\phi(x, y)$ is a formula and f is a function such that $\forall_\mu^* \sigma M_\sigma \models \exists x \phi[x, f(\sigma)]$. We show that $M \models \exists x \phi[x, [f]_\mu]$.

Let $g(\sigma) = \{x \in \sigma \mid (\exists y \in OD(\mu, x))(M_\sigma \models \phi[y, f(\sigma)])\}$. Then $\forall_\mu^* \sigma g(\sigma)$ is a non-empty subset of σ . By normality of μ , there is a fixed real x such that $\forall_\mu^* \sigma x \in g(\sigma)$. Hence we can define $h(\sigma)$ to be the least y in $OD(\mu, x)$ such that $M_\sigma \models \phi[y, f(\sigma)]$. It’s easy to see then that $M \models \phi[[h]_\mu, [f]_\mu]$. \square

By Lemma 2.3, M and H are well-founded so we identify them with their transitive collapse. First note that $M \models ZF + DC$ and $H \models ZFC$. We then observe that $\Omega = [\lambda\sigma.\omega_1]_\mu$ is measurable in M and in H . This is because ω_1 is measurable in M_σ and H_σ for all σ . Note also that $\Omega > \Theta$ as $\forall_\mu^* \sigma$, Θ^{M_σ} is countable and $\mathcal{P}(\omega_1)^{M_\sigma}$ is countable. The key for this is the easily verified fact: There are no sequences of ω_1^V distinct reals. Hence, by a standard Vopenka argument, for any set of ordinals $A \in M$ of size less than Ω , there is an H -generic G_A (for a forcing of size smaller than Ω) such that $A \in H[G_A] \subseteq M$ and Ω is also measurable in $H[G_A]$.

Lemma 2.4. $\mathcal{P}(\mathbb{R}) \subseteq M$.

Proof. Let $A \subseteq \mathbb{R}$. Then there is an $x \in \mathbb{R}$ such that $A \in OD(x, \mu)$. By fineness of μ , $(\forall_\mu^* \sigma)(x \in \sigma)$ and hence $(\forall_\mu^* \sigma)(A \cap \sigma \in OD(x, \mu, \sigma))$. So we have $(\forall_\mu^* \sigma)(A \cap \sigma \in M_\sigma)$. This gives us that $A = [\lambda\sigma.A \cap \sigma]_\mu \in M$. \square

Lemma 2.4 implies that M contains all bounded subsets of Θ . Now we're ready to run our core model induction.

Theorem 2.5. *PD holds.*

Proof. We will show that $M_n^\#$ exists for all n . This will imply PD. We will in fact show that in M , the $M_n^\#$ operators are total on H_Ω . We need to do this since in our argument, we'll try to build K (in H and its generic extensions) up to Ω and need to know that this can be done (i.e. K exists and is $\Omega + 1$ -iterable in the appropriate model).

To start off, it's easy to see that in M , the $\#$ -operator is total on H_Ω . This is because Ω is measurable in M . The same conclusion holds for H and any generic extension J of H by a forcing of size smaller than Ω and $J \subseteq M$.

We want to show that in M , the $M_1^\#$ -operator is total on H_Ω . We do this in two steps. First, we prove

Lemma 2.6. *For each $x \in \mathbb{R}$, $M_1^\#(x)$ exists.*

Proof. This is the key lemma. For brevity, we just show $M_1^\#$ exists. The proof relativizes trivially to any real. Suppose not. Then in H , K (built up to Ω) exists and is $\Omega + 1$ iterable. This is because in H , the $\#$ -operator is total on H_Ω . Let $\kappa = \omega_1$. By Lemma 2.2, κ is inaccessible in H and in any set generic extension J of H and $J \subseteq M$. By [11], $K^H = K^{H[G]}$ for any H -generic G for a poset of size smaller than Ω . We use K to denote K^H .

Claim. $(\kappa^+)^K = (\kappa^+)^H$.

Proof. The proof follows that of Theorem 3.1 in [5]. Suppose not. Let $\lambda = (\kappa^+)^K$. Hence $\lambda < (\kappa^+)^H$. Working in H , let N be a transitive, power admissible set such that ${}^\omega N \subseteq N$, $V_\kappa \cup \mathcal{J}_{\lambda+1}^K \subseteq N$, and $\text{card}(N) = \kappa$. We then choose $A \subseteq \kappa$ such that $N \in L[A]$ and $K^{L[A]}|_\lambda = K|_\lambda$ and $\lambda = (\kappa^+)^{K^{L[A]}}$. Such an A exists by Lemma 3.1.1 in [5] and the fact that $\lambda < (\kappa^+)^H$.

Now, since $A^\#$ exists in H , $(\kappa^+)^{L[A]} < (\kappa^+)^H$. By *GCH* in $L[A]$, $\text{card}^H(\mathcal{P}(\kappa) \cap L[A]) = \kappa$. So in M , there is an $L[A]$ -ultrafilter U over κ that is nonprincipal and countably complete (in M and in V). This is because such a U exists in V as being induced from μ and since U can be coded as a subset of ω_1 , $U \in M$. Let J be a generic extension of H (of size smaller than Ω) such that $U \in J$. From now on, we work in J . Let

$$j : L[A] \rightarrow \text{Ult}(L[A], U) = L[j(A)]$$

be the ultrapower map. Then j is well-founded, $\text{crit}(j) = \kappa$, $A = j(A) \cap \kappa \in L[j(A)]$. So $L[A] \subseteq L[j(A)]$. The key point here is that $\mathcal{P}(\kappa) \cap K^{L[A]} = \mathcal{P}(\kappa) \cap K^{L[j(A)]}$. To see this, first note that the \subseteq direction holds because any κ -strong mouse in $L[A]$ is a κ -strong mouse in $L[j(A)]$ as $\mathbb{R} \cap L[A] = \mathbb{R} \cap L[j(A)]$ and $L[A]$ and $L[j(A)]$ have the same $< \kappa$ -strong mice. To see the converse, suppose not. Then there is a sound mouse $\mathcal{M} \triangleleft K^{L[j(A)]}$ such that \mathcal{M} extends $K^{L[A]}|_\lambda$ and \mathcal{M} projects to κ . Such an \mathcal{M} exists by a theorem of Ralf Schindler which essentially states that K is just a stack of mice above ω_2 (here $\omega_2^{L[j(A)]} < \kappa$). The iterability of \mathcal{M} is absolute between J and $L[j(A)]$, hence $\mathcal{M} \triangleleft K$ by the following folklore result

Lemma 2.7. *Assume $ZFC +$ “there is no model with a Woodin.” Let M be a transitive model that satisfies $ZFC^- +$ “there is no inner model of a Woodin”. Furthermore, assume that $\omega_1 \subseteq M$. Let $\mathcal{P} \in M$ be a premouse with no definable Woodin. Then*

$$\mathcal{P} \text{ is a mouse} \Leftrightarrow M \models \mathcal{P} \text{ is a mouse}.$$

For a proof of this, see [6]. So $\mathcal{M} \triangleleft K|_\lambda = K^{L[A]}|_\lambda$. This is a contradiction.

Now the rest of the proof is just as in that of Theorem 3.1 in [5]. Let E_j be the superstrong extender derived from j . Since $\text{card}(N) = \kappa$ and $\lambda < \kappa^+$, a standard argument (due to Kunen) shows that $F, G \in L[j(A)]$ where

$$F = E_j \cap ([j(\kappa)]^{<\omega} \times K^{L[A]})$$

and,

$$G = E_j \cap ([j(\kappa)]^{<\omega} \times N).$$

This is because F and G have size κ in $L[j(A)]$. So $(K^{L[j(A)]}, F)$ and (N, G) are elements of $L[j(A)]$. In $L[j(A)]$, for cofinally many $\xi < j(\kappa)$, $F \restriction \xi$ coheres with K and (N, G) is a weak \mathcal{A} -certificate for $(K, F \restriction \xi)$ (in the sense of [5]), where

$$\mathcal{A} = \bigcup_{n < \omega} \mathcal{P}([\kappa]^n)^K.$$

By Theorem 2.3 in [5], those segments of F are on the extender sequence of $K^{L[j(A)]}$. But then κ is Shelah in $K^{L[j(A)]}$, which is a contradiction. \square

The proof of the claim also shows that $(\kappa^+)^K = (\kappa^+)^J$ for any set (of size smaller than Ω) generic extension J of H . In particular, since any $A \subseteq \omega_1$ belongs to a set generic extension of H of size smaller than Ω , we immediately get that $(\kappa^+)^K = \omega_2$. This is impossible in the presence of μ . To see this, let $\vec{C} = \langle C_\alpha \mid \alpha < \omega_2 \rangle$ be the canonical \square_κ -sequence in K . Working in V , let ν be the measure on $\mathcal{P}_{\omega_1}(\omega_2)$ induced by μ defined as follows. First, fix a surjection $\pi : \mathbb{R} \rightarrow \omega_2$. Then π trivially induces a surjection from $\mathcal{P}_{\omega_1}(\mathbb{R})$ onto $\mathcal{P}_{\omega_1}(\omega_2)$ which we also call π . Note that π is well-defined because $\text{cof}(\omega_2^V) > \omega$. Then our measure ν is defined as

$$A \in \nu \Leftrightarrow \pi^{-1}[A] \in \mu.$$

Now consider the ultrapower map $j : K \rightarrow \text{Ult}(K, \nu) = K^*$. An easy calculation gives us that $j''\omega_2 = [\lambda\sigma.\sigma]_\nu$ and $A \in \nu \Leftrightarrow j''\omega_2 \in j(A)$. So let $\gamma = j''\omega_2$ and $\vec{D} = j(\vec{C}) \in K^*$. Note that $(\kappa^+)^{K^*} = \omega_2$ and since $K^* \models ZFC$, ω_2 is regular in K^* . Also $\gamma < j(\omega_2^V)$. Now consider the set D_γ . By definition, D_γ is an ω -club in γ so it has order type at least ω_2 . However, let $C = \langle \alpha < \omega_2 \mid \text{cof}(\alpha) = \omega \rangle$. Then $j(C) = j''C$ is an ω -club in γ . Hence $E = D_\gamma \cap j(C)$ is an ω -club in γ . For each $\alpha \in \lim E \subseteq E$, $D_\alpha = D_\gamma \cap \alpha$ and D_α has order type less than ω_1 . This implies that every initial segment of D_γ has order type less than ω_1 which is a contradiction. \square

The lemma shows that the $M_1^\#$ -operator is total on H_κ (in M as well as in V). This implies that $\forall_\mu^* \sigma$, M_σ is closed under the $M_1^\#$ -operator on H_κ . By L6s, M is closed under the $M_1^\#$ -operator on H_Ω . Similar conclusions hold for H as well as its generic extensions in M .

Now to handle the case $M_n^\# \Rightarrow M_{n+1}^\#$ (for $n > 0$), we run the same argument as above except for a few modifications in the proof of Lemma 2.6. First, we replace $L[A]$ in the proof of Lemma 2.6 by $L^{M_n^\#}[A] \restriction \Omega$ (this is because the $M_n^\#$ -operator is a function from H_Ω to H_Ω). Next, we build a version of K , call it $K^{M_n^\#}$, that is closed under the $M_n^\#$ -operator; one way to do this is in the K^c construction, at the successor step, we replace the *rud* operator by the $M_n^\#$ operator. Finally, we replace Lemma 2.7 by the following

Lemma 2.8. *Suppose $\forall x(M_n^\# \text{ exists})$ and “there is no inner model with $n + 1$ Woodins.” Assume M is a model that satisfies $ZFC^- + \forall x(M_n^\# \text{ exists}) +$ “there is no inner model with $n + 1$ Woodins.” Assume also $\omega_1^V \subseteq M$. Let $\mathcal{P} \in M$ be a premouse closed under the $M_n^\#$ -operator. Then*

$$\mathcal{P} \text{ is a mouse} \Leftrightarrow M \models \mathcal{P} \text{ is a mouse.}$$

These modifications allow us to conclude again that $(\kappa^+)^{K^{M_n^\#}} = (\kappa^+)^H$ as in the claim of Lemma 2.6. Then, repeating the argument after the claim, we get a contradiction (the \square_κ -sequence can still be constructed in $K^{M_n^\#}$ since the $M_n^\#$ -operator has condensation). \square

Theorem 2.9. $AD^{L(\mathbb{R})}$ holds.

Proof. The core model induction through $L(\mathbb{R})$ is guided by the pattern of scales in [10]. The notations we use in this proof are completely standard. All of the key concepts being used in this proof are nicely summarized on pages 2-5 of [9] and we’ll use the same notations as that paper. To show $AD^{L(\mathbb{R})}$, we show $L(\mathbb{R}) \models \forall \alpha W_\alpha^*$. Our plan is to show $W_{\alpha+1}^*$ assuming W_α^* for α critical. Theorem 2.5 provides the base case for our induction. For $\alpha > 0$, we have three cases:

1. α is a successor of a critical ordinal or α is a limit of critical ordinals and $\text{cof}(\alpha) = \omega$;
2. α is inadmissible, limit of critical ordinals, $\text{cof}(\alpha) > \omega$
3. α ends a weak gap or successor of an ordinal that ends a strong gap.

We deal with the easy case (case 1) first. In this case, let $\Gamma = \Sigma_1(\mathcal{J}_\alpha(\mathbb{R}))$. Then $C_\Gamma = \bigcup_{n < \omega} C_{\Gamma_n}$ for some increasing sequence of scaled pointclasses $\langle \Gamma_n \mid n < \omega \rangle$. By W_α^* , for each n , we have mouse operators $\langle J_m^n \mid m < \omega \rangle$ (each J_m^n is total on H_Ω^M) that collectively witness AD^{Γ_n} . The desired mouse operator J_0 is defined as follows: For each transitive and self-wellordered $A \in H_\Omega^M$, $J_0(A)$ is the shortest initial segment $\mathcal{M} \triangleleft Lp(A)$ such that $\mathcal{M} \models ZFC^-$ and \mathcal{M} is closed under J_n^m for all m, n . J_0 is total and trivially relativizes well because the J_n^m ’s are total and relativize well. We then use the proof of Lemma 2.6 to get that J_1 is total on H_{ω_1} and use μ to lift the operator J_1 to H_Ω^M . Similarly, we get that J_n is total on H_Ω^M for all n . By Lemma 4.1.3 of [7], this implies $W_{\alpha+1}^*$.

Now we’re on to the case where α is inadmissible and $\text{cof}(\alpha) > \omega$. Let $\phi(v_0, v_1)$ be a Σ_1 formula and $x \in \mathbb{R}$ be such that

$$\forall y \in \mathbb{R} \exists \beta < \alpha \mathcal{J}_\beta(\mathbb{R}) \models \phi[x, y],$$

and letting $\beta(x, y)$ be the least such β ,

$$\alpha = \sup\{\beta(x, y) \mid y \in \mathbb{R}\}.$$

We first define J_0 on transitive and self-wellordered $A \in H_{\omega_1}$ coding x . For $n < \omega$, let

$$\phi^*(n) \equiv \exists \gamma (\mathcal{J}_\gamma(\mathbb{R}) \models \forall i \in \omega (i > 0 \Rightarrow \phi((v)_0, (v)_1) \wedge (\gamma + \omega n) \text{ exists})).$$

For such an A as above, let \mathcal{M} be an A -premouse and G be a $Col(\omega, A)$ -generic over \mathcal{M} , then $\mathcal{M}[G]$ can be regarded as a $z(G, A)$ -mouse where $z(G, A)$ is a real coding G, A and is obtained from G, A in some simple fashion. Also, let σ_A be a term defined uniformly (in \mathcal{M}) from A, x such that

$$(\sigma_A^G)_0 = x$$

and

$$\{(\sigma_A^G)_i \mid i > 0\} = \{\rho^G \mid \rho \in L_1(A) \wedge \rho^G \in \mathbb{R}\}.$$

Let φ be a sentence in the language of A -premise such that for any A -premouse \mathcal{M} , $\mathcal{M} \models \varphi$ iff whenever G is \mathcal{M} -generic for $Col(\omega, A)$, then for any n there is a $\gamma < o(\mathcal{M})$ such that

$$\mathcal{M}[z(G, A)] \restriction \gamma \text{ is a } \langle \phi_n^*, \rho_A^G \rangle\text{-prewitness.}$$

Then $J_0(A)$ is the shortest initial segment of $Lp(A)$ which satisfies φ , if it exists, and is undefined otherwise. Clearly, $J_0(A)$ exists for all $A \in H_{\omega_1}$ coding x because α has uncountable cofinality and there are only countably many $\langle \phi_n^*, \rho_A^G \rangle$. By Lemma 4.2.3 in [7], J_0 relativizes well. We can then use μ to lift the domain of J_0 from H_{ω_1} to H_Ω^M just as in the proof of Theorem 2.5. Also we can show J_n is total on H_Ω^M for all n . This implies $W_{\alpha+1}^*$.

Lastly, we consider the gap case. Let $[\bar{\alpha}, \alpha^*]$ be the gap where $\alpha^* = \alpha$ if the gap is weak and $\alpha^* = \alpha - 1$ if the gap is strong. Let $\Gamma = \Sigma_1(\mathcal{J}_{\bar{\alpha}}(\mathbb{R}))$ and \mathcal{N} be a Γ -suitable mouse with ω_1 -iteration strategy Σ . We may assume Σ has condensation as we may take Σ to be the strategy guided by a sjs \vec{A} that seals the gap. We again use μ to extend Σ to a Ω iteration strategy (in M). Of course, the point is that Σ is $OD_{\mu, x}$ for some $x \in \mathbb{R}$, hence

$$\forall_\mu^* \sigma (\mathcal{N} \in M_\sigma \wedge \Sigma \restriction M_\sigma \in M_\sigma).$$

Now, we'll define a hybrid mouse operator J_0 on $A \in H_\Omega^M$ and A codes \mathcal{N} . $J_0(A)$ is an ordinary sharp for $L^\Sigma(A)$ (where $L^\Sigma(A)$ is built up to Ω). We again get that J_n is total on H_Ω^M by the same argument as that in Theorem 2.5 with only one modification: the core

model K (built in an appropriate universe) is the hybrid core model i.e. K is closed under Σ and an appropriate formulation of Lemma 2.8. By Lemma 5.6.8 in [7], we obtain $W_{\alpha+1}^*$. \square

The same proof gives us $AD^{K(\mathbb{R})}$. What we need to carry out the core induction through $K(\mathbb{R})$ is the scales analysis of $K(\mathbb{R})$ much like that of $L(\mathbb{R})$. The analysis in [13] and the following theorem (which generalizes the main theorem of [8]) give us the pattern of scales needed for the core model induction.

Theorem 2.10. *Let \mathcal{M} be a countably ω -iterable mouse over $\mathbb{R}^{\mathcal{M}}$ which satisfies AD and suppose $[\alpha, \beta]$ is a weak gap of \mathcal{M} (β may be $o(\mathcal{M})$). Suppose there is a strategy Σ such that $\mathcal{M} \models$ “ Σ has branch condensation” and $\Gamma =_{\text{def}} \Sigma_1^{\mathcal{M}|\alpha}$ is captured by Σ -mice with iteration strategy in $\mathcal{M}|\alpha$, i.e., $\forall b \in HC^{\mathcal{M}}(Lp^{\Sigma, \Gamma}(b) \cap \mathcal{P}(b) = C_{\Gamma}(b))$. Then we have that \mathcal{M} believes that $\Sigma_n^{\mathcal{M}|\beta}$ has the scale property where n is least such that $\rho_n(\mathcal{M}|\beta) = \mathbb{R}^{\mathcal{M}}$.*

Let

$$\Gamma_0 = \{A \subseteq \mathbb{R} \mid L(A, \mathbb{R}) \models AD + \Theta = \theta_0\},$$

and $M_0 = L(\Gamma_0, \mathbb{R})$. Note that if $A_0, A_1 \in \Gamma_0$, then they are Wadge comparable. This is because there are no divergent models of AD^+ in V . To see this, suppose not then by an (unpublished) result of Woodin, there is a model $M \subseteq L(\mathbb{R}, \mu)$ containing $\mathbb{R} \cup ORD$ such that $M \models AD_{\mathbb{R}}$. Let ν be the Solovay measure in M and $\rho = \mu \upharpoonright M$. By results of Woodin (see [14]), $\rho = \nu$. This implies $L(\mathbb{R}, \rho) \subseteq M$ and $L(\mathbb{R}, \rho) \models “AD + \rho$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})”$. Since $L(\mathbb{R}, \rho)$ is not a model of $AD_{\mathbb{R}}$, $L(\mathbb{R}, \rho) \subsetneq M \subseteq L(\mathbb{R}, \mu)$. This contradicts that $L(\mathbb{R}, \mu) = L(\mathbb{R}, \rho)$. By [4], for any $A \in \Gamma_0$,

$$L(A, \mathbb{R}) \models MSC.$$

This implies that, by Theorem 17.1 in [12],

$$L(A, \mathbb{R}) \models V = K(\mathbb{R}).$$

These facts immediately give us the following

Lemma 2.11. $M_0 = K(\mathbb{R})$ and $M_0 \models AD + \Theta = \theta_0$.

Remark. $AD^{K(\mathbb{R})}$ is the most amount of determinacy one could hope to prove. This is because if μ comes from the Solovay measure (derived from winning strategies of real games) in an $AD^+ + AD_{\mathbb{R}} + SMSC$ universe, call it V (any $AD_{\mathbb{R}} + V = L(\mathcal{P}(\mathbb{R}))$ -model below $AD_{\mathbb{R}} + \Theta$ is regular would do here), then $L(\mathbb{R}, \mu)^V \cap \mathcal{P}(\mathbb{R}) \subseteq K(\mathbb{R})^V$. This is because μ is OD hence $\mathcal{P}(\mathbb{R}) \cap L(\mu, \mathbb{R}) \subseteq \mathcal{P}_{\theta_0}(\mathbb{R})$. Since $AD^+ + SMC$ gives us that any set of reals of Wadge rank $< \theta_0$

is contained in an \mathbb{R} -mouse, we get that $\mathcal{P}(\mathbb{R}) \cap L(\mu, \mathbb{R}) \subseteq K(\mathbb{R})$ (it is conceivable that the inclusion is strict). By Theorem 1.1, $L(\mathbb{R}, \mu) \models \Theta = \theta_0$, which implies $L(\mathbb{R}, \mu) \models V = K(\mathbb{R})$. Putting all of this together, we get $L(\mathbb{R}, \mu) \models K(\mathbb{R}) = L(\mathcal{P}(\mathbb{R})) + AD^{K(\mathbb{R})}$.

The above remark suggests that we should try to show that every set of reals in $V = L(\mathbb{R}, \mu)$ is captured by an \mathbb{R} -mouse, which will prove Theorem 1.2. This is accomplished in the next two sections.

3 $\Theta^{K(\mathbb{R})} = \Theta$

Suppose for contradiction that $\Theta^{K(\mathbb{R})} < \Theta$. For simplicity, we first get a contradiction from the smallness assumption that “there is no model containing $\mathbb{R} \cup ORD$ that satisfies $AD^+ + \Theta > \theta_0$ ”. The argument will closely follow the argument in Chapter 7 of [7]. All of our key notions and notations come from there unless specified otherwise. Let $\Theta^* = \Theta^{K(\mathbb{R})}$. Let \mathcal{M}_∞ be $HOD^{K(\mathbb{R})} \restriction \Theta^*$. Then $\mathcal{M}_\infty = \mathcal{M}_\infty^+ \restriction \Theta^*$ where \mathcal{M}_∞^+ is the limit of a directed system (the hod limit system) indexed by pairs (\mathcal{P}, \vec{A}) where \mathcal{P} is a suitable premouse, \vec{A} is a finite sequence of OD sets of reals, and \mathcal{P} is strongly \vec{A} -quasi-iterable in $K(\mathbb{R})$. For more details on how the direct limit system is defined, the reader should consult Chapter 7 of [7]. Let Γ be the collection of $OD^{K(\mathbb{R})}$ sets of reals. For each $\sigma \in \mathcal{P}_{\omega_1}(\mathbb{R})$ such that $Lp(\sigma) \models AD^+$, let $\mathcal{M}_\infty^\sigma$ and Γ^σ be defined the same as \mathcal{M}_∞ and Γ but in $Lp(\sigma)$. Let $\Theta^\sigma = o(\mathcal{M}_\infty^\sigma)$. By $AD^{K(\mathbb{R})}$ and $\Theta^* < \Theta$, we easily get

Lemma 3.1. $\forall_\mu^* \sigma (Lp(\sigma) \models AD^+, \text{ and there is an elementary map } \pi_\sigma : (Lp(\sigma), \mathcal{M}_\infty^\sigma, \Gamma^\sigma) \rightarrow (K(\mathbb{R}), \mathcal{M}_\infty, \Gamma)).$

Proof. First, it's easily seen that $K(\mathbb{R}) \models AD^+$ implies $\forall_\mu^* \sigma Lp(\sigma) \models AD^+$. We also have that letting ν be the induced measure on $\mathcal{P}_{\omega_1}(K(\mathbb{R}))$

$$\forall_\nu^* X \ X \prec K(\mathbb{R}).$$

The second clause of the lemma follows by transitive collapsing the X 's above. Note that $\forall_\mu^* \sigma Lp(\sigma)$ is the uncollapse of some countable $X \prec K(\mathbb{R})$ such that $\mathbb{R}^X = \sigma$. This is because if \mathcal{M} is an \mathbb{R} -mouse then $\forall_\nu^* X \ \mathcal{M} \in X$. The π_σ 's are just the uncollapse maps. \square

We may as well assume $(\forall_\mu^* \sigma)(Lp(\sigma) = Lp(\sigma)^{K(\mathbb{R})})$ as otherwise, fix a σ such that $Lp(\sigma) \models AD^+$ and $\mathcal{M} \triangleleft Lp(\sigma)$ a sound mouse over σ , $\rho_\omega(\mathcal{M}) = \sigma$ and $\mathcal{M} \notin Lp(\sigma)^{K(\mathbb{R})}$. Let Λ be the strategy of \mathcal{M} . Then by a core model induction as above, we can show that $L^\Lambda(\mathbb{R}) \models AD^+ + \Theta > \theta_0$. Since this is very similar to the proof of PD , we only mention a few key points for this induction. First, Λ is a $\omega_1 + 1$ strategy with condensation and $\forall_\mu^* \sigma \ \Lambda \restriction M_\sigma \in M_\sigma$ and

$\forall_\mu^* \sigma \ \Lambda \restriction H_\sigma[\mathcal{M}] \in H_\sigma[\mathcal{M}]$. This allows us to lift Λ to a $\Omega + 1$ strategy in M and construct K^Λ up to Ω inside $\prod_\sigma H_\sigma[\mathcal{M}]$. This is a contradiction to our smallness assumption.

Lemma 3.2. $\forall_\mu^* \sigma \ \mathcal{M}_\infty^\sigma$ is full in $K(\mathbb{R})$ in the sense that $Lp(\mathcal{M}_\infty^\sigma) \models \Theta^\sigma$ is Woodin.

Proof. First note that $Lp_2(\sigma) =_{def} Lp(Lp(\sigma)) \models AD^+ + \Theta = \theta_0$ because $\mathcal{P}(\mathbb{R})^{Lp_2(\sigma)} = \mathcal{P}(\mathbb{R})^{Lp(\sigma)}$. So suppose $\mathcal{N}^\sigma \triangleright \mathcal{M}_\infty^\sigma$ is the Q -structure. It's easy to see that $\mathcal{N}^\sigma \in Lp_2(\sigma)$ and is in fact OD there.

Next we observe that in $Lp_2(\sigma)$, $\Theta = \Theta^\sigma$. By a Theorem of Woodin, we know $HOD^{Lp_2(\sigma)} \models \Theta^\sigma$ is Woodin (see Theorem 5.6 of [3]). But this is a contradiction to our assumption that \mathcal{N}^σ is a Q -structure for Θ^σ . \square

The last lemma shows that for a typical σ , $Lp_\omega(\mathcal{M}_\infty^\sigma)$ is suitable in $K(\mathbb{R})$. Let $\mathcal{M}_\infty^{\sigma,+}$ be the hod limit computed in $Lp(\sigma)$. Let $(\Gamma^\sigma)^{<\omega} = \{\vec{A}_n \mid n < \omega\}$ and for each $n < \omega$, let \mathcal{N}_n be such that \mathcal{N}_n is strongly \vec{A}_n -quasi-iterable in $Lp(\sigma)$ such that $\mathcal{M}_\infty^{\sigma,+}$ is the quasi-limit of the \mathcal{N}_n 's in $Lp(\sigma)$. Let $\mathcal{M}_\infty^{\sigma,*}$ be the quasi-limit of the \mathcal{N}_n 's in $K(\mathbb{R})$. We'll show that $\pi_\sigma''\Gamma^\sigma$ is cofinal in Γ , $\mathcal{M}_\infty^{\sigma,+} = \mathcal{M}_\infty^{\sigma,*} = Lp_\omega(\mathcal{M}_\infty^\sigma)$ and hence $\mathcal{M}_\infty^{\sigma,+}$ is strongly A -quasi-iterable in $K(\mathbb{R})$ for each $A \in \pi_\sigma''\Gamma^\sigma$. From this we'll get a strategy Σ_σ for $\mathcal{M}_\infty^{\sigma,+}$ with weak condensation. This proceeds much like the proof in Chapter 7 of [7].

Let T be the tree for a universal $(\Sigma_1^2)^{K(\mathbb{R})}$ -set; let $T^* = \prod_\sigma T$ and $T^{**} = \prod_\sigma T^*$. To show $(\forall_\mu^* \sigma)(\pi_\sigma''\Gamma^\sigma \text{ is cofinal in } \Gamma)$ we first observe that

$$(\forall_\mu^* \sigma)(L[T^*, \mathcal{M}_\infty^\sigma] \restriction \Theta^\sigma = \mathcal{M}_\infty^\sigma),$$

that is, T^* does not create Q -structures for $\mathcal{M}_\infty^\sigma$. This is because $\mathcal{M}_\infty^\sigma$ is countable, ω_1^V is inaccessible in any inner model of choice, $L[T^*, \mathcal{M}_\infty^\sigma] \restriction \omega_1^V = L[T, \mathcal{M}_\infty^\sigma] \restriction \omega_1^V$, and $L[T, \mathcal{M}_\infty^\sigma] \restriction \Theta^\sigma = \mathcal{M}_\infty^\sigma$ by Lemma 3.2. Next, let E_σ be the extender derived from π_σ with generators in $[\gamma]^{<\omega}$, where $\gamma = \sup \pi_\sigma''\Theta^\sigma$. By the above, E_σ is a pre-extender over $L[T^*, \mathcal{M}_\infty^\sigma]$.

Lemma 3.3. $(\forall_\mu^* \sigma)(Ult(L[T^*, \mathcal{M}_\infty^\sigma], E_\sigma) \text{ is wellfounded})$.

Proof. The statement of the lemma is equivalent to

$$Ult(L[T^{**}, \mathcal{M}_\infty], \prod_\sigma E_\sigma) \text{ is wellfounded. } (*)$$

To see $(*)$, note that

$$\prod_\sigma E_\sigma = E_\mu$$

where E_μ is the extender from the ultrapower map j_μ by μ (with generators in $[\xi]^{<\omega}$, where $\xi = \sup j_\mu''\Theta^*$). This uses normality of μ . We should mention that the equality above should

be interpreted as saying: the embedding by $\prod_{\sigma} E_{\sigma}$ agrees with j_{μ} on all ordinals (less than Θ).

Since μ is countably complete and DC holds, we have that $Ult(L[T^{**}, \mathcal{M}_{\infty}], E_{\mu})$ is well-founded. Hence we're done. \square

Theorem 3.4. 1. $(\forall_{\mu}^* \sigma)(\pi_{\sigma} \text{ is continuous at } \theta^{\sigma}).$ Hence $\text{cof}(\Theta^{K(\mathbb{R})}) = \omega$.

2. If $i : \mathcal{M}_{\infty}^{\sigma} \rightarrow S$, and $j : S \rightarrow \mathcal{M}_{\infty}$ are elementary and $\pi_{\sigma} = j \circ i$ and S is countable in $K(\mathbb{R})$, then S is full in $K(\mathbb{R})$. In fact, if W is the collapse of a hull of S containing $\text{rng}(i)$, then W is full in $K(\mathbb{R})$.

Proof. The keys are Lemma 3.3 and the fact that the tree T^* , which enforces fullness for \mathbb{R} -mice, does not generate Q -structures for $\mathcal{M}_{\infty}^{\sigma}$. To see (1), suppose not. Fix a typical σ for which (1) fails. Let $\gamma = \sup \pi_{\sigma}'' \Theta^{\sigma} < \Theta^*$. Let E_{σ} be the extender derived from π_{σ} with generators in $[\gamma]^{<\omega}$ and consider the ultrapower map

$$\tau : L[T^*, \mathcal{M}_{\infty}^{\sigma}] \rightarrow N_{\sigma} =_{\text{def}} Ult(L[T^*, \mathcal{M}_{\infty}^{\sigma}], E_{\sigma}).$$

We may as well assume N_{σ} is transitive by Lemma 3.3. We have that τ is continuous at Θ^{σ} and $N_{\sigma} \models o(\tau(\mathcal{M}_{\infty}^{\sigma}))$ is Woodin. Since $o(\tau(\mathcal{M}_{\infty}^{\sigma})) = \gamma < \Theta^*$, there is a Q -structure \mathcal{Q} for $o(\tau(\mathcal{M}_{\infty}^{\sigma}))$ in $K(\mathbb{R})$. But \mathcal{Q} can be constructed from T^* , hence from $\tau(T^*)$. To see this, suppose $\mathcal{Q} = \prod_{\sigma} \mathcal{Q}_{\sigma}$ and $\gamma = \prod_{\sigma} \gamma_{\sigma}$. Then $\forall_{\mu}^* \sigma \mathcal{Q}_{\sigma}$ is the Q -structure for $\mathcal{M}_{\infty}^{\sigma} | \gamma_{\sigma}$ and the iterability of \mathcal{Q}_{σ} is certified by T . This implies the iterability of \mathcal{Q} is certified by T^* . But $\tau(T^*) \in N_{\sigma}$, which does not have Q -structures for $\tau(\mathcal{M}_{\infty}^{\sigma})$. Contradiction.

(1) shows then that $\pi_{\sigma}'' \Gamma^{\sigma}$ is cofinal in Γ . The proof of (2) is similar. We just prove the first statement of (2). The point is that i can be lifted to an elementary map

$$i^* : L[T^*, \mathcal{M}_{\infty}^{\sigma}] \rightarrow L[\overline{T}, S]$$

for some \overline{T} and j can be lifted to

$$j^* : L[\overline{T}, S] \rightarrow N_{\sigma}$$

by the following definition

$$j^*(i^*(f)(a)) = \tau(f)(j(a))$$

for $f \in L[T^*, \mathcal{M}_{\infty}^{\sigma}]$ and $a \in [o(\mathcal{S})]^{<\omega}$. By the same argument as above, \overline{T} certifies iterability of mice in $K(\mathbb{R})$ and hence enforces fullness for S in $K(\mathbb{R})$. This is what we want. \square

We can define a map $\tau : \mathcal{M}_\infty^{\sigma,+} \rightarrow \mathcal{M}_\infty^{\sigma,*}$ as follows. Let $x \in \mathcal{M}_\infty^{\sigma,+}$. There is an $i < \omega$ and a y such that in $Lp(\sigma)$, $x = \pi_{\mathcal{N}_i, \infty}^{A_i}(y)$, where $\pi_{\mathcal{N}_i, \infty}^{A_i}$ is the direct limit map from $H_{A_i}^{\mathcal{N}_i}$ into $\mathcal{M}_\infty^{\sigma,+}$ in $Lp(\sigma)$. Let

$$\tau(x) = \pi_{\mathcal{N}_i, \mathcal{M}_\infty^{\sigma,*}}^{A_i}(y),$$

where $\pi_{\mathcal{N}_i, \mathcal{M}_\infty^{\sigma,*}}^{A_i}$ witnesses $(\mathcal{N}_i, A_i) \preceq (\mathcal{M}_\infty^{\sigma,*}, A_i)$ in the hod direct limit system in $K(\mathbb{R})$.

Lemma 3.5. 1. $\mathcal{M}_\infty^{\sigma,*} = H_{\pi''\Gamma_\sigma}^{\mathcal{M}_\infty^{\sigma,*}}$; furthermore, for any quasi-iterate \mathcal{Q} of $\mathcal{M}_\infty^{\sigma,*}$, $\mathcal{Q} = H_{\pi''\Gamma_\sigma}^{\mathcal{Q}}$ and $\pi_{\mathcal{M}_\infty^{\sigma,*}, \mathcal{Q}}^{\pi''\Gamma_\sigma}(\tau_A^{\mathcal{M}_\infty^{\sigma,+}}) = \tau_A^{\mathcal{Q}}$ for all $A \in \pi''\Gamma_\sigma$.

2. $\tau = id$ and $\mathcal{M}_\infty^{\sigma,+} = \mathcal{M}_\infty^{\sigma,*}$.

3. $\pi_\sigma = \pi_{\mathcal{M}_\infty^{\sigma,+}, \infty}^{\pi''\Gamma_\sigma}$.

Proof. The proof is just that of Lemmata 7.8.7 and 7.8.8 in [7]. We first show (1). In this proof, “suitable” means suitable in $K(\mathbb{R})$. The key is for any quasi-iterate \mathcal{Q} of $\mathcal{M}_\infty^{\sigma,*}$, we have

$$\pi_\sigma|_{\mathcal{M}_\infty^{\sigma,+}} = \pi_{\mathcal{Q}, \infty}^{\pi''\Gamma_\sigma} \circ \pi_{\mathcal{M}_\infty^{\sigma,*}, \mathcal{Q}}^{\pi''\Gamma_\sigma} \circ \tau. \quad (*)$$

Using this and Theorem 3.4, we get $H_{\pi''\Gamma_\sigma}^{\mathcal{Q}} = \mathcal{Q}$ for any quasi-iterate \mathcal{Q} of $\mathcal{M}_\infty^{\sigma,*}$. To see this, first note that \mathcal{Q} is suitable; Theorem 3.4 implies the collapse \mathcal{S} of $H_{\pi''\Gamma_\sigma}^{\mathcal{Q}}$ must be suitable. This means, letting δ be the Woodin of \mathcal{Q} , $H_{\pi''\Gamma_\sigma}^{\mathcal{Q}}|(\delta+1) = \mathcal{Q}|(\delta+1)$. Next, we show $H_{\pi''\Gamma_\sigma}^{\mathcal{Q}}|((\delta^+)^{\mathcal{Q}}) = \mathcal{Q}|((\delta^+)^{\mathcal{Q}})$. The proof of this is essentially that of Lemma 4.35 in [2]. We sketch the proof here. Suppose not. Let $\pi : \mathcal{S} \rightarrow \mathcal{Q}$ be the uncollapse map. Note that $\text{crt}(\pi) = (\delta^+)^{\mathcal{S}}$ and $\pi((\delta^+)^{\mathcal{S}}) = (\delta^+)^{\mathcal{Q}}$. Let \mathcal{R} be the result of first moving the least measurable of $\mathcal{Q}|((\delta^+)^{\mathcal{Q}})$ above δ and then doing the genericity iteration (inside \mathcal{Q}) of the resulting model to make $\mathcal{Q}|\delta$ generic at the Woodin of \mathcal{R} . Let \mathcal{T} be the resulting tree. Then \mathcal{T} is maximal with $lh(\mathcal{T}) = (\delta^+)^{\mathcal{Q}}$; $\mathcal{R} = Lp(\mathcal{M}(\mathcal{T}))$; and the Woodin of \mathcal{R} is $(\delta^+)^{\mathcal{Q}}$. Since $\{\gamma_A^{\mathcal{R}} \mid A \in \pi''\Gamma_\sigma\}$ are definable from $\{\tau_{A, (\delta^+)^{\mathcal{Q}}}^{\mathcal{Q}} \mid A \in \pi''\Gamma_\sigma\}$, they are in $\text{rng}(\pi)$. This gives us that $\sup H_{\pi''\Gamma_\sigma}^{\mathcal{Q}} \cap (\delta^+)^{\mathcal{Q}} = (\delta^+)^{\mathcal{Q}}$, which easily implies $(\delta^+)^{\mathcal{Q}} \subseteq H_{\pi''\Gamma_\sigma}^{\mathcal{Q}}$. The proof that $(\delta^{+n})^{\mathcal{Q}} \subseteq H_{\pi''\Gamma_\sigma}^{\mathcal{Q}}$ for $1 < n < \omega$ is similar and is left for the reader.

(2) easily follows from (1). (3) follows using (*) and $\tau = id$. \square

For each σ such that Theorem 3.4 and Lemma 3.5 hold for σ , let Σ_σ be the canonical strategy for $\mathcal{M}_\infty^\sigma$ as guided by $\pi''\Gamma_\sigma$. Recall $\pi''\Gamma_\sigma$ is a cofinal collection of $OD^{K(\mathbb{R})}$ sets of reals. The existence of Σ_σ follows from Theorem 7.8.9 in [7]. Note that Σ_σ has weak condensation, i.e., suppose \mathcal{Q} is a Σ_σ iterate of $\mathcal{M}_\infty^{\sigma,+}$ and $i : \mathcal{M}_\infty^{\sigma,+} \rightarrow \mathcal{Q}$ is the iteration map, and suppose $j : \mathcal{M}_\infty^{\sigma,+} \rightarrow \mathcal{R}$ and $k : \mathcal{R} \rightarrow \mathcal{Q}$ are such that $i = k \circ j$ then \mathcal{R} is suitable (in the sense of $K(\mathbb{R})$).

Definition 3.6 (Branch condensation). *Let $\mathcal{M}_\infty^{\sigma,+}$ and Σ_σ be as above. We say that Σ_σ has branch condensation if for any Σ_σ iterate \mathcal{Q} of $\mathcal{M}_\infty^{\sigma,+}$, letting $k : \mathcal{M}_\infty^{\sigma,+} \rightarrow \mathcal{Q}$ be the iteration map, for any maximal tree \mathcal{T} on $\mathcal{M}_\infty^{\sigma,+}$, for any cofinal non-dropping branch b of \mathcal{T} , letting $i = i_b^\mathcal{T}$, $j : \mathcal{M}_b^\mathcal{T} \rightarrow \mathcal{P}$, where \mathcal{P} is a Σ_σ iterate of $\mathcal{M}_\infty^\sigma$ with iteration embedding k , suppose $k = j \circ i$, then $b = \Sigma_\sigma(\mathcal{T})$.*

Theorem 3.7. $(\forall_\mu^* \sigma)(A \text{ tail of } \Sigma_\sigma \text{ has branch condensation.})$

Proof. The proof is like that of Theorem 7.9.1 in [7]. We only mention the key points here. We assume that $\forall_\mu^* \sigma$ no Σ_σ -tails have branch condensation. Fix such a σ . First, let $X_\sigma = \text{rng}(\pi_\sigma \upharpoonright \mathcal{M}_\infty^{\sigma,+})$ and

$$H = HOD_{\{\mu, \mathcal{M}_\infty^{\sigma,+}, \mathcal{M}_\infty, \pi_\sigma, T^*, X_\sigma, x_\sigma\}},$$

where x_σ is a real enumerating $\mathcal{M}_\infty^{\sigma,+}$. So $H \models ZFC + “\mathcal{M}_\infty^\sigma \text{ is countable and } \omega_1^V \text{ is measurable.}”$

Next, let \overline{H} be a collapse of a countable elementary substructure of a sufficiently large rank-initial segment of H . Let $(\gamma, \rho, \mathcal{N}, \nu)$ be the preimage of $(\omega_1^V, \pi_\sigma, \mathcal{M}_\infty, \mu)$ under the uncollapse map, call it π . We have that $\overline{H} \models ZFC^- + “\gamma \text{ is a measurable cardinal as witnessed by } \nu.”$ This \overline{H} will replace the countable iterable structure obtained from the hypothesis HI(c) in Chapter 7 of [7]. Now, in $K(\mathbb{R})$, the following hold true:

1. There is a term $\tau \in \overline{H}$ such that whenever g is a generic over \overline{H} for $Col(\omega, < \gamma)$, then τ^g is a $(\rho, \mathcal{M}_\infty^{\sigma,+}, \mathcal{N})$ -certified bad sequence. See Definitions 7.9.3 and 7.9.4 in [7] for the notions of a bad sequence and a $(\rho, \mathcal{M}_\infty^{\sigma,+}, \mathcal{N})$ -certified bad sequence respectively.
2. Whenever $i : \overline{H} \rightarrow J$ is a countable linear iteration map by the measure ν and g is J -generic for $Col(\omega, < i(\gamma))$, then $i(\tau)^g$ is truly a bad sequence.

The proof of (1) and (2) is just like that of Lemma 7.9.7 in [7]. The key is that in (1), any $(\rho, \mathcal{M}_\infty^{\sigma,+}, \mathcal{N})$ -certified bad sequence is truly a bad sequence from the point of view of $K(\mathbb{R})$ and in (2), any countable linear iterate J of \overline{H} can be realized back into H by a map ψ in such a way that $\pi = \psi \circ i$.

Finally, using (1), (2), the iterability of \overline{H} , and an AD^+ -reflection in $K(\mathbb{R})$ like that in Theorem 7.9.1 in [7], we get a contradiction. \square

Theorem 3.7 allows us to run the core model induction in $L(\Sigma_\sigma, \mathbb{R})$ and show that $L(\Sigma_\sigma, \mathbb{R}) \models AD$. This along with the fact that $\Sigma_\sigma \notin K(\mathbb{R})$ imply

$$L(\Sigma_\sigma, \mathbb{R}) \models \Theta > \theta_0.$$

This is a contradiction to our smallness assumption.

4 AD in $L(\mathbb{R}, \mu)$

Now we know $\Theta^{K(\mathbb{R})} = \Theta$. We want to show $\mathcal{P}(\mathbb{R}) \cap K(\mathbb{R}) = \mathcal{P}(\mathbb{R})$.

Lemma 4.1. $\mathcal{P}(\mathbb{R}) \cap L(T^*, \mathbb{R}) = \mathcal{P}(\mathbb{R}) \cap K(\mathbb{R})$.

Proof. By MC in $K(\mathbb{R})$, we have

$$(\forall_\mu^* \sigma)(\mathcal{P}(\sigma) \cap L(T, \sigma) = Lp(\sigma) \cap \mathcal{P}(\sigma)).$$

This proves the lemma. \square

We now show that μ is amenable to $K(\mathbb{R})$ in the sense that μ restricting to any Wadge initial segment of $\mathcal{P}(\mathbb{R})^{K(\mathbb{R})}$ is in $K(\mathbb{R})$. The following lemma is due to Woodin.

Lemma 4.2. *Suppose $S = \{(x, A_x) \mid x \in \mathbb{R} \wedge A_x \in \mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))\} \in K(\mathbb{R})$. Then $\mu \upharpoonright S = \{(x, A_x) \mid \mu(A_x) = 1\} \in K(\mathbb{R})$.*

Proof. Let A_S be an ∞ -Borel code⁵ for S in $K(\mathbb{R})$. We may pick A_S such that it is a bounded subset of Θ^* . We may as well assume that A_S is $OD^{K(\mathbb{R})}$ and A_S codes T . This gives us

$$(\forall_\mu^* \sigma)(\mathcal{P}(\sigma) \cap L(A_S, \sigma) = \mathcal{P}(\sigma) \cap L(T, \sigma)),$$

or equivalently letting $A_S^* = \prod_\sigma A_S$,

$$\mathcal{P}(\mathbb{R}) \cap L(A_S^*, \mathbb{R}) = L(T^*, \mathbb{R}).$$

We have the following equivalences:

$$\begin{aligned} (x, A_x) \in \mu \upharpoonright S &\Leftrightarrow (\forall_\mu^* \sigma)(\sigma \in A_x \cap \mathcal{P}_{\omega_1}(\sigma)) \\ &\Leftrightarrow (\forall_\mu^* \sigma)(L(A_S, \sigma) \models \emptyset \Vdash_{Col(\omega, \sigma)} \sigma \in A_x \cap \mathcal{P}_{\omega_1}(\sigma)) \\ &\Leftrightarrow L(A_S^*, \mathbb{R}) \models \emptyset \Vdash_{Col(\omega, \mathbb{R})} \mathbb{R} \in A_x. \end{aligned}$$

The above equivalences show that $\mu \upharpoonright S \in L(S^*, \mathbb{R})$. But by Lemma 4.1 and the fact that $\mu \upharpoonright S$ can be coded as a set of reals in $L(S^*, \mathbb{R})$, hence $\mu \upharpoonright S \in L(T^*, \mathbb{R})$, we have that $\mu \upharpoonright S \in K(\mathbb{R})$. \square

Lemma 4.3. $\mathcal{P}(\mathbb{R}) \cap K(\mathbb{R}) = \mathcal{P}(\mathbb{R})$. Hence $L(\mathbb{R}, \mu) \models AD$.

⁵If $S \subseteq \mathbb{R}$, A_S is an ∞ -Borel code for S if $A_S = (T, \psi)$ where T is a set of ordinals and ψ is a formula such that for all $x \in \mathbb{R}$, $x \in S \Leftrightarrow L[T, x] \models \psi[T, x]$.

Proof. First we observe that if α is such that there is a new set of reals in $L_{\alpha+1}(\mathbb{R})[\mu] \setminus L_\alpha(\mathbb{R})[\mu]$ then there is a surjection from \mathbb{R} onto $L_\alpha(\mathbb{R})[\mu]$. This is because the predicate μ is a predicate for a subset of $\mathcal{P}(\mathbb{R})$, which collapses to itself under collapsing of hulls of $L_\alpha(\mathbb{R})[\mu]$ that contain all reals. With this observation, the usual proof of condensation (for L) goes through with one modification: one must put all reals into hulls one takes.

Now suppose for a contradiction that there is an $A \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}, \mu)$ such that $A \notin K(\mathbb{R})$. Let α be least such that $A \in L_{\alpha+1}(\mathbb{R})[\mu] \setminus L_\alpha(\mathbb{R})[\mu]$. We may assume that $\mathcal{P}(\mathbb{R}) \cap L_\alpha(\mathbb{R})[\mu] \subseteq K(\mathbb{R})$. By the above observation, $\alpha < \Theta = \Theta^{K(\mathbb{R})}$ because otherwise, there is a surjection from \mathbb{R} on Θ , which contradicts the definition of Θ . Now if $\mathcal{P}(\mathbb{R}) \cap L_\alpha(\mathbb{R})[\mu] \subsetneq \mathcal{P}(\mathbb{R}) \cap K(\mathbb{R})$, then by Lemma 4.2, $\mu \upharpoonright \mathcal{P}(\mathbb{R}) \cap L_\alpha(\mathbb{R})[\mu] \in K(\mathbb{R})$. But this means $A \in K(\mathbb{R})$. So we may assume $\mathcal{P}(\mathbb{R}) \cap L_\alpha(\mathbb{R})[\mu] = \mathcal{P}(\mathbb{R}) \cap K(\mathbb{R})$. But this means that we can in $L_\Theta(\mathbb{R})[\mu]$ use $\mu \upharpoonright \mathcal{P}(\mathbb{R}) \cap L_\alpha(\mathbb{R})[\mu]$ compute $\Theta^K(\mathbb{R})$ and this contradicts the fact that $\Theta^{K(\mathbb{R})} = \Theta$. \square

Lemma 4.3 along with Theorem 1.1 imply Theorem 1.2 assuming the smallness assumption in the previous section. We now show how to get rid of it.

Recall that we have shown $AD^{K(\mathbb{R})}$. The proof of this section shows that if $\Theta^{K(\mathbb{R})} = \Theta$ then $L(\mathbb{R}, \mu) \models AD$, which proves Theorem 1.2. So suppose $\Theta > \Theta^{K(\mathbb{R})}$. Then the proof of the previous section produces a strategy Σ with branch condensation. By a similar core model induction to that of $AD^{K(\mathbb{R})}$, we get $AD^{K^\Sigma(\mathbb{R})}$ ⁶. Now $K^\Sigma(\mathbb{R})$ is the maximal model of $AD^+ + \Theta = \theta_1$.

Let $M = K^\Sigma(\mathbb{R})$ and $H = HOD_{\mathbb{R}}^M$. Note that $\mathcal{P}(\mathbb{R})^H = \mathcal{P}(\mathbb{R})^{K(\mathbb{R})} = \mathcal{P}_{\theta_0}(\mathbb{R})^M$. We aim to show that $L(\mathbb{R}, \mu) \subseteq H$, which is a contradiction. By the proof of Theorem 4.2, we get that $\nu =_{\text{def}} \mu \upharpoonright \mathcal{P}(\mathbb{R})^H \in M$. Let $\pi : \mathbb{R}^\omega \rightarrow \mathcal{P}_{\omega_1}(\mathbb{R})$ be the canonical map, i.e. $\pi(\vec{x}) = \text{rng}(\vec{x})$. Let $A \subseteq \mathcal{P}_{\omega_1}(\mathbb{R})$ be in H . There is a natural interpretation of A as a set of Wadge rank less than θ_0^M , that is the preimage \vec{A} of A under π has Wadge rank less than θ_0^M . Fix such an A ; note that \vec{A} is invariant in the sense that whenever $\vec{x} \in \vec{A}$ and $\vec{y} \in \mathbb{R}^\omega$ and $\text{rng}(\vec{x}) = \text{rng}(\vec{y})$ then $\vec{y} \in \vec{A}$. Let $G_{\vec{A}}$ and G_A be the real games corresponding to \vec{A} and A respectively. Suppose $\langle x_i \mid i < \omega \rangle \in \mathbb{R}^\omega$ is a typical play in either game, then the payoff is as follows:

Player I wins the play in $G_{\vec{A}}$ if $\langle x_i \mid i < \omega \rangle \in \vec{A}$,

and

Player I wins the play in G_A if $\{x_i \mid i < \omega\} \in A$.

Lemma 4.4. G_A is determined.

⁶See [4] for the definition of $K^\Sigma(\mathbb{R})$. The extra ingredient needed for the core model induction through $K^\Sigma(\mathbb{R})$ is the theory of hod mice developed in [4].

Proof. For each $\vec{x} \in \mathbb{R}^\omega$, let $\sigma_{\vec{x}} = \text{rng}(\vec{x})$. Consider the games $G_{\vec{A}}^{\vec{x}}$ and $G_A^{\sigma_{\vec{x}}}$ which have the same rules and payoffs as those of $G_{\vec{A}}$ and G_A respectively except that players are required to play reals in $\sigma_{\vec{x}}$. Note that these games are determined and Player I wins the game $G_{\vec{A}}^{\vec{x}}$ iff Player I wins the corresponding game $G_A^{\sigma_{\vec{x}}}$.

Without loss of generality, suppose $\nu(\{\sigma \in \mathcal{P}_{\omega_1}(\mathbb{R}) \mid \text{Player I wins } G_A^\sigma\}) = 1$. For each such σ , let τ_σ be the canonical winning strategy for Player I given by the Moschovakis's Third Periodicity Theorem. We can easily integrate these strategies to construct a strategy τ for Player I in G_A . We show how to define $\tau(\emptyset)$ and it'll be clear that the definition of τ on finite sequences is similar. Let ρ be the restriction of μ on the Suslin co-Suslin sets of M . Note that $\rho \in M$. We know

$$\forall_\rho^* \sigma \ \tau_\sigma(\emptyset) \in \sigma.$$

We have to use ρ since the set displayed above in general does not have Wadge rank less than θ_0 in M . Normality of ρ implies

$$\exists x \in \mathbb{R} \ \forall_\rho^* \sigma \ \tau_\sigma(\emptyset) = x.$$

Let $\tau(\emptyset) = x$ where x is as above. It's easy to show τ is a winning strategy for Player I in G_A . \square

The lemma and standard results of Woodin (see [14]) show that ρ is the unique normal fine measure on the Suslin co-Suslin sets of M and hence $\rho \in OD^M$. This means $\rho \restriction \mathcal{P}(\mathbb{R})^H = \nu$ is OD in M . This implies $L(\mathbb{R}, \nu) \subseteq H$. But $L(\mathbb{R}, \nu) = L(\mathbb{R}, \mu)$. Contradiction.⁷

5 Open problems and questions

We first mention the following

Conjecture: Suppose $L(\mathbb{R}) \models DC + \Theta$ is inaccessible. Then $L(\mathbb{R}) \models AD$.

This is arguably the analogous statement in $L(\mathbb{R})$ of our main theorem. It is tempting to conjecture that if $L(\mathbb{R}) \models DC + \Theta > \omega_2$ then $L(\mathbb{R}) \models AD$ but this is known to be false by theorems of Harrington [1]. Next, we mention the following uniqueness problem which

⁷Another proof of this is through inner model theory. The existence of M gives us $\mathcal{M}_{\omega_2}^\sharp$. We can use the canonical strategy of $\mathcal{M}_{\omega_2}^\sharp$ to do an \mathbb{R} -genericity iteration in such a way that if the end model is \mathcal{N} and λ is the sup of \mathcal{N} 's Woodin cardinals then there is a generic filter g over \mathcal{N} for $\text{Col}(\omega, < \lambda)$ and a filter F defined in $\mathcal{N}[g]$ such that $L(\mathbb{R}, F) \models ZF + AD + F$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$. Furthermore, $\mathcal{P}(\mathbb{R})^{L(\mathbb{R}, F)}$ is projective in Σ , hence is Suslin co-Suslin in M . The proof above gives us a contradiction.

concerns the relationship between AD models of the form $L(\mathbb{R}, \mu)$.

Open problem: Suppose $L(\mathbb{R}, \mu_i) \models “ZF + DC + AD + \mu_i \text{ is a normal fine measure on } \mathcal{P}_{\omega_1}(\mathbb{R})”$ for $i = 0, 1$. Must $L(\mathbb{R}, \mu_0) = L(\mathbb{R}, \mu_1)$?

We suspect that the answer is no but haven’t been able to construct two distinct models of the form $L(\mathbb{R}, \mu)$ that satisfy AD . By Theorem 1.1, if $L(\mathbb{R}, \mu_0)$ and $L(\mathbb{R}, \mu_1)$ are the same model then $\mu_0 \cap L(\mathbb{R}, \mu_0) = \mu_1 \cap L(\mathbb{R}, \mu_1)$. A generalization of the problem proved in this paper is to consider determinacy in models of the form $L(S, \mathbb{R}, \mu)$ where S is a set of ordinals and $L(S, \mathbb{R}, \mu) \models “ZF + DC + \Theta > \omega_2 + \mu \text{ is a normal fine measure on } \mathcal{P}_{\omega_1}(\mathbb{R})”$. Here is a (vague) conjecture.

Conjecture: Let $L(S, \mathbb{R}, \mu)$ be as above. Let $\Theta = \Theta^{L(S, \mathbb{R}, \mu)}$ and M_∞ be the maximal model of determinacy in $L(S, \mathbb{R}, \mu)$. Then either $\Theta^{M_\infty} = \Theta$ or there is a model of “ $AD_{\mathbb{R}} + \Theta$ is regular” containing $\mathbb{R} \cup ORD$.

In another direction, we could ask about how to identify the first stage in the core model induction (under appropriate hypotheses) that reaches $AD^{L(\mathbb{R}, \mu)}$ where μ comes from some filter on $\mathcal{P}_{\omega_1}(\mathbb{R})$ and $L(\mathbb{R}, \mu) \models “\mu \text{ is a normal fine measure on } \mathcal{P}_{\omega_1}(\mathbb{R})”$. A problem of this kind is the following

Open problem: Suppose I_{NS} is saturated and $WRP_2^*(\omega_2)$ ⁸. Must there be a filter μ on $\mathcal{P}_{\omega_1}(\mathbb{R})$ such that $L(\mathbb{R}, \mu) \models “AD + \mu \text{ is a normal fine measure on } \mathcal{P}_{\omega_1}(\mathbb{R})”$?

This problem is discussed in [9] though in a slightly different formulation. The point is that the hypothesis of the problem is obtained in a \mathbb{P}_{\max} -extension of a model of the form $L(\mathbb{R}, \mu) \models “AD + \mu \text{ is a normal fine measure on } \mathcal{P}_{\omega_1}(\mathbb{R})”$.

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⁸ I_{NS} is the nonstationary ideal on ω_1 and $WRP_2^*(\omega_2)$ is defined in section 9.5 of [15].

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